Student Workbook

Decomposing Rational Expressions

Name: ______________________ Date: ____________ Class: ________

ESSENTIAL QUESTION

How does partial fraction decomposition reverse the process of adding rational expressions, and why is this technique essential for integration in calculus?

Lesson Overview

Partial fraction decomposition is the process of writing a rational expression P(x)/Q(x) as a sum of simpler fractions — it is the reverse of adding rational expressions. Prerequisite: the degree of the numerator must be LESS than the degree of the denominator. If not, perform polynomial long division first.

Four Cases + Two Methods to Find the Constants

Case 1	Distinct linear factors (x−a)(x−b)⋯: use A/(x−a) + B/(x−b) + ⋯
Case 2	Repeated linear factor $(x-a)^n$: use $A_1/(x-a) + A_2/(x-a)^2 + \cdots + A_n/(x-a)^n$.
Case 3	Irreducible quadratic ($b^2-4ac < 0$): use $(Ax+B)/(ax^2+bx+c)$.
Case 4	Repeated irreducible quadratic: use $(A_1x+B_1)/Q + (A_2x+B_2)/Q^2 + \cdots$
Methods	Method 1 — plug in roots of Q(x) to solve directly. Method 2 — coefficient matching: expand both sides and equate like powers of x.

Worked Examples

EXAMPLE 1

Decompose: (3x+5) / ((x+1)(x+2))

- Set up: A/(x+1) + B/(x+2). Multiply by (x+1)(x+2): 3x+5 = A(x+2) + B(x+1)
- x = −1: 3(−1)+5 = A(1) → A = 2
- x = −2: 3(−2)+5 = B(−1) → B = 1

Answer: 2/(x+1) + 1/(x+2)

EXAMPLE 2

Decompose: $(2x^2 + 3x - 1) / ((x-1)(x+1)(x+2))$

- Set up: A/(x−1) + B/(x+1) + C/(x+2)
- Multiply: $2x^2+3x-1 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$
- x=1: 4=6A→A=2/3. x=−1: −2=−2B→B=1. x=−2: 1=3C→C=1/3

Answer: (2/3)/(x−1) + 1/(x+1) + (1/3)/(x+2)

EXAMPLE 3

Decompose: $(x+3) / (x(x-1)^2)$

- Set up (repeated linear factor): $A/x + B/(x-1) + C/(x-1)^2$
- Multiply: $x+3 = A(x-1)^2 + Bx(x-1) + Cx$
- $x=0$: $3=A$. $x=1$: $4=C$. Match x^2 coeff: $0=A+B \rightarrow B=-3$

Answer: $3/x - 3/(x-1) + 4/(x-1)^2$

EXAMPLE 4

Decompose: $(2x+1) / ((x^2+1)(x-1))$

- Set up (irreducible quadratic + linear): $(Ax+B)/(x^2+1) + C/(x-1)$
- Multiply: $2x+1 = (Ax+B)(x-1) + C(x^2+1)$. $x=1$: $3=2C \rightarrow C=3/2$
- Match x^2: $A+C=0 \rightarrow A=-3/2$. Match x^1: $-A+B=2 \rightarrow B=1/2$. Check x^0: $-B+C=1$ ✓

Answer: $(-3x/2 + 1/2)/(x^2+1) + (3/2)/(x-1)$

EXAMPLE 5

Decompose: $(x^2+2x+3) / ((x+1)(x^2+2x+2))$

- x^2+2x+2 is irreducible (discriminant $= 4-8 = -4 < 0$)
- Set up: $A/(x+1) + (Bx+C)/(x^2+2x+2)$. Multiply: $x^2+2x+3 = A(x^2+2x+2) + (Bx+C)(x+1)$
- $x=-1$: $2=A$. Match x^2: $A+B=1 \rightarrow B=-1$. Match x^1: $2A+B+C=2 \rightarrow C=-1$. Check x^0: $2A+C=3$ ✓

Answer: $2/(x+1) + (-x-1)/(x^2+2x+2)$

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Decompose: $(5x+1) / ((x+1)(x-1))$

Hint: Set up $A/(x+1) + B/(x-1)$, multiply through, and plug in $x = -1$ and $x = 1$.

2 Decompose: $(3x+7) / ((x+2)^2)$

Hint: Repeated linear factor — use $A/(x+2) + B/(x+2)^2$.

3 Decompose: $(x^2+1) / (x(x+1)(x-2))$

Hint: Three distinct linear factors — use $A/x + B/(x+1) + C/(x-2)$.

4 Decompose: $(2x+3) / ((x^2+1)(x+1))$

Hint: One irreducible quadratic and one linear factor — use $(Ax+B)/(x^2+1) + C/(x+1)$.

5 Decompose: $(x^3+2x^2+x+1) / (x(x^2+1))$

Hint: Degree of numerator = degree of denominator — do polynomial long division first, then decompose the remainder.

Key Vocabulary

Partial Fraction Decomp.	Writing $P(x)/Q(x)$ as a sum of simpler rational expressions with lower-degree denominators. Ex: $(3x+5)/((x+1)(x+2)) = 2/(x+1)+1/(x+2)$.
Proper Expression	A rational expression where the degree of the numerator is strictly less than the degree of the denominator.
Improper Expression	Degree of numerator ≥ degree of denominator. Requires polynomial long division before decomposing.
Repeated Linear Factor	A factor $(x-a)^n$ with multiplicity $n > 1$. Requires $A_1/(x-a) + A_2/(x-a)^2 + \cdots + A_n/(x-a)^n$.
Irreducible Quadratic	A quadratic ax^2+bx+c with discriminant $b^2-4ac < 0$ — cannot be factored over the reals. Requires a linear numerator $Ax+B$.
Coefficient Matching	Expanding both sides and equating coefficients of like powers of x to form a system of equations for the unknown constants.

Check Your Understanding

Circle the letter of the correct answer for each question.

1 **The partial fraction form of $A/((x+1)(x-2))$ uses:** Multiple Choice

A $A/((x+1)(x-2))$

B $A/(x+1) + B/(x-2)$

C $(Ax+B)/(x+1) + C/(x-2)$

D $A/(x+1)^2 + B/(x-2)$

2 **Before decomposing $(x^3+1)/(x^2+x)$, you must first:** Multiple Choice

A Factor the denominator

B Do polynomial long division

C Multiply by the denominator

D Find the roots

3 **The partial fraction form for $1/((x-1)^2(x+2))$ is:** Multiple Choice

- **A** $A/(x-1)^2 + B/(x+2)$
- **B** $A/(x-1) + B/(x-1)^2 + C/(x+2)$
- **C** $(Ax+B)/(x-1)^2 + C/(x+2)$
- **D** $A/(x-1) + B/(x+2)$

4 **For an irreducible quadratic factor (x^2+4), the partial fraction term is:** Multiple Choice

- **A** $A/(x^2+4)$
- **B** $(Ax+B)/(x^2+4)$
- **C** $A/x + B/4$
- **D** $Ax/(x^2+4)$

5 **Decompose $1/((x+1)(x-1))$. The constants A and B are:** Multiple Choice

- **A** $A=1$, $B=1$
- **B** $A=1/2$, $B=-1/2$
- **C** $A=-1/2$, $B=1/2$
- **D** $A=1$, $B=-1$

Common Mistakes to Avoid

✗ Using a constant numerator A for an irreducible quadratic factor.

✓ Irreducible quadratic factors (x^2+bx+c) require a LINEAR numerator $Ax+B$, not just A.

✗ Forgetting to include all powers for repeated factors.

✓ For $(x-a)^2$, you need BOTH $A/(x-a)$ AND $B/(x-a)^2$. Include every power from 1 up to n.

✗ Trying to decompose an improper fraction directly.

✓ If degree(numerator) ≥ degree(denominator), do polynomial long division FIRST, then decompose the remainder.

✗ Only using the 'plug in roots' method and missing constants for irreducible quadratics.

✓ For irreducible quadratic factors, plug in roots won't work (they're complex). Use coefficient matching instead.

Math Tips

★ The 'plug in roots' method is the fastest way to find constants for linear factors. For each factor (x−a), plug in x=a.

★ After finding constants by plugging in roots, use coefficient matching to find any remaining constants (for irreducible quadratic factors).

★ Always check your answer by recombining the partial fractions — you should get back the original expression.

★ The number of unknown constants equals the degree of the denominator (after factoring). This is a useful check.

★ Partial fractions are the key technique for integrating rational functions in calculus. Mastering this now pays dividends later.